A SINGLE STABILITY PARAMETER FOR LINEAR 2-PORT CIRCUITS

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ABSTRACT

A new stability parameter, "μ(S)," is defined for linear 2-port circuits. It is shown that $\mu > 1$ alone is necessary and sufficient for the circuit to be unconditionally stable. This single parameter can replace the traditional Rollet condition K > 1 which requires that an additional auxiliary condition also be met for absolute stability.

1. INTRODUCTION

A linear 2-port circuit is said to be absolutely, or unconditionally, stable if there is no passive source, $|\Gamma_S| < 1$, and passive load, $|\Gamma_L| < 1$, combination that can cause the circuit to oscillate. It has been shown [9], [2], [3] that the combination of the Rollett [7] condition

$$K = \frac{1-|S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|} > 1 \tag{1}$$

together with *any one* of the following auxiliary conditions is necessary and sufficient for unconditional stability.

$$B_1 = 1+|S_{11}|^2 - |S_{22}|^2 - |\Delta|^2 > 0 \tag{2a}$$

$$B_2 = 1-|S_{11}|^2 + |S_{22}|^2 - |\Delta|^2 > 0 \tag{2b}$$

$$|\Delta| = |S_{11}S_{22} - S_{12}S_{21}| < 1 \tag{2c}$$

$$1-|S_{11}|^2 > |S_{12}S_{21}| \tag{2d}$$

$$1-|S_{22}|^2 > |S_{12}S_{21}| \tag{2e}$$

The design of active circuits requires that multiple parameters be evaluated over a wide frequency range much larger than their intended pass-band. If a circuit or a device fails to meet these conditions, it is difficult to assess the degree of potential instability that exists since the values associated with (1) and (2) provide little direct physical insight into the degree of stability or lack thereof.

A New Approach

The input and output reflection coefficient are related to the load and source reflection coefficient by the well known linear fractional transformation which maps circles into circles (where a straight line is the special case of a circle containing the point ∞) [12]

$$\Gamma_{in} = f(\Gamma_L) = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1-S_{22}\Gamma_L}, \tag{3a}$$

and

$$\Gamma_{out} = g(\Gamma_s) = S_{22} + \frac{S_{12}S_{21}\Gamma_s}{1-S_{11}\Gamma_s}. \tag{3b}$$

The inverses, $\Gamma_L = f^{-1}(\Gamma_{in})$, and $\Gamma_S = g^{-1}(\Gamma_{out})$, are well defined provided that $S_{12}S_{21} \neq 0$

$$\Gamma_L = f^{-1}(\Gamma_{in}) = \frac{S_{11} - \Gamma_{in}}{\Delta - S_{22}\Gamma_{in}} \tag{3c}$$

$$\Gamma_S = g^{-1}(\Gamma_{out}) = \frac{S_{22} - \Gamma_{out}}{\Delta - S_{11}\Gamma_{out}}. \tag{3d}$$

Consequently, the approach in this paper is to initially assume that the circuit is not unilateral and then to examine the unilateral case afterwards.

A circuit is unconditionally stable if the function "f" maps the unit disk in the Γ_L-plane into the unit disk in the Γ_{in}-plane (see figure 1). This is equivalent to saying that the inverse f^{-1} maps the unit disk in the Γ_{in}-plane onto a region which contains the unit disk in the Γ_L-plane. Note that the unit disk is a set of complex reflection coefficient whose magnitude is less than one. This is exactly the region represented by the conventional or passive Smith Chart denoted in this paper as USC standing for *Unit Smith Chart*. Because of the circle preserving property of linear fractional transformations the inverse mapping could typically look either like figure 1b or 1c. These functional characteristics and their analytical representation form the basis for defining the new measure of stability.

A new parameter, "μ," will be defined based upon the mapping "f." It will be shown that the value of μ alone, unambiguously determines if the circuit is unconditionally stable or potentially unstable. A dual parameter designated μ' can be defined based upon the mapping "g" and it also uniquely determines whether the circuit is unconditionally stable. This approach also provides direct physical insight

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into the degree to which the Unit Smith Chart, USC, is encroached by possible unstable loads and source regions providing the engineer with a measure of the risk or margin associated with his design.

Complex Representation of a Disk (or Disk Complement)

The following inequality

$$|z|^2 - za - z^* a^* < b \qquad (4)$$

where a is a complex number and b is a real number such that $|a|^2 + b \geq 0$, describes a circular disk of complex points whose center is $C = a^*$, $|C| = c$, and whose radius is

$$r = \sqrt{b + |a|^2}.$$

This is seen by adding the term $|a|^2$ to both sides of (4) and manipulating the results to get

$$|z - a^*| < \sqrt{b + |a|^2}.$$

If the "less than" sign in (4) were reversed to be a "greater than" sign then

$$|z - a^*| > \sqrt{b + |a|^2}$$

which describes a region external to the above defined disk. This external region is referred to therefore as a "disk complement."

2. DEFINING THE NEW STABILITY FACTOR μ

The new parameter, μ, is defined as the minimum distance in the Γ_L-plane between the origin of the Unit Smith Chart USC and the unstable region. A negative value for this miss distance parameter indicates that the unstable region overlaps the origin of USC. It turns out that μ is described by a relatively simple expression whose analytical form is the same regardless of whether the inverse mapping, f^{-1}, is of the type illustrated by figure 1b, or figure 1c.

It will now be shown that the mapping "f^{-1}," illustrated in figures (1b) and (1c), will occur if and only if the distance $\mu(S) > 1$. This will be argued by showing that these mappings imply that $\mu(S) > 1$ and then justifying the reversibility of the steps. The above statement is equivalent to the following mathematical statement

$$\left[USC \subset \left\{\Gamma_L : \Gamma_L = f^{-1}(|\Gamma_{in}| < 1)\right\}\right] \Leftrightarrow \mu(S) > 1. \qquad (5)$$

The range of the map $f^{-1}(|\Gamma_{in}| < 1)$, is determined by $|f(\Gamma_L)| < 1$. Straight forward substitution from (3a) yields

$$|\Gamma_L|^2\left[|S_{22}|^2 - |\Delta|^2\right] + \Gamma_L^*\left[S_{11}\Delta^* - S_{22}^*\right] + \Gamma_L\left[S_{11}^*\Delta - S_{22}\right] > |S_{11}|^2 - 1.$$

Dividing this expression by $|S_{22}|^2 - |\Delta|^2$, one obtains the complex variable representation of a disc or disc complement (see (4)) depending on whether or not

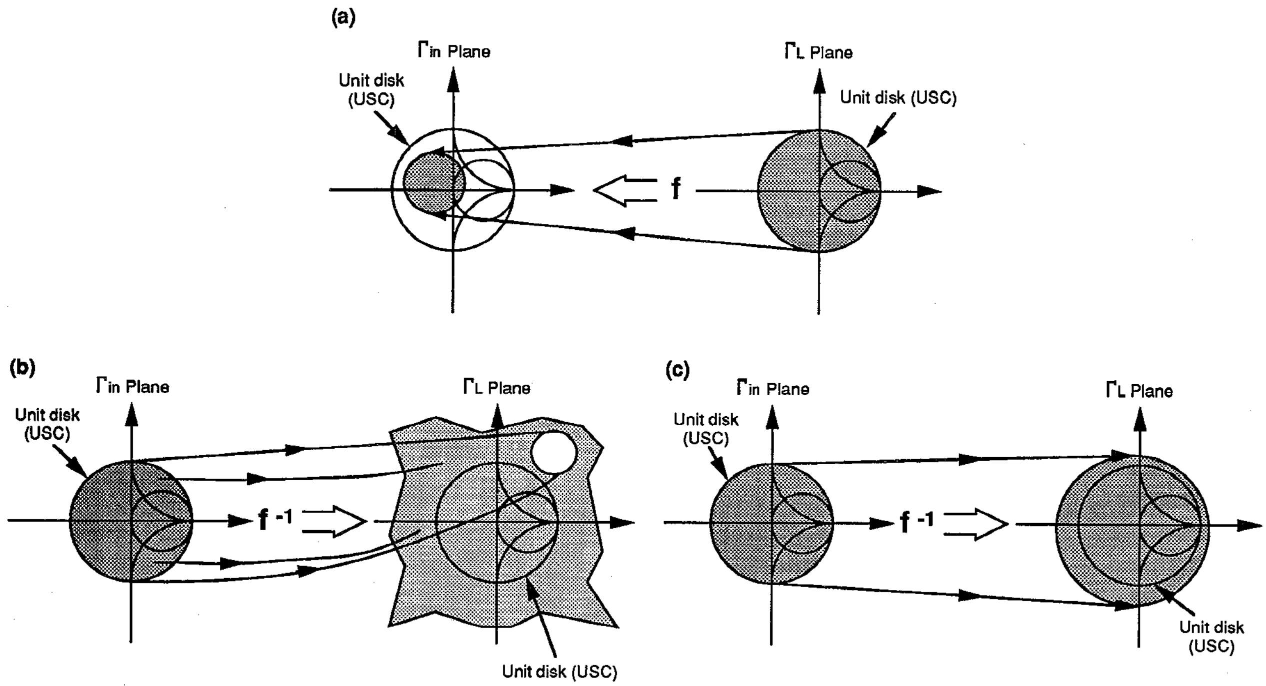

Figure 1. Unconditional Stability in Terms of Mapped Regions

$|S_{22}|^2 - |\Delta|^2 < 0$ or $|S_{22}|^2 - |\Delta|^2 > 0$. The resulting inequalities are as follows,

$$\left|\Gamma_L - \frac{S_{22}^* - S_{11}\Delta^*}{|S_{22}|^2 - |\Delta|^2}\right| < \frac{|S_{21}S_{12}|}{|\Delta|^2 - |S_{22}|^2} \tag{6}$$

where $|S_{22}|^2 - |\Delta|^2 < 0$
and

$$\left|\Gamma_L - \frac{S_{22}^* - S_{11}\Delta^*}{|S_{22}|^2 - |\Delta|^2}\right| > \frac{|S_{21}S_{12}|}{|S_{22}|^2 - |\Delta|^2} \tag{7}$$

where $|S_{22}|^2 - |\Delta|^2 > 0$.

The circle defined by replacing the inequalities in (6) and (7) with an **equal sign** is commonly referred to as a stability circle. The formulation resulting in (6) and (7) preserves the information about which region is the stable one.

One must now consider what is required for the USC to be contained in the range of the mapping f^{-1} as illustrated in figures 1b and 1c.

Case 1: $|S_{22}|^2 - |\Delta|^2 > 0$

In this case the range of our mapping in the Γ_L plane is the region outside the circle defined by (7) and must be of the type illustrated in figure 1b. It is clear that USC is contained in the locus of points if and only if

$$c - r > 1 \tag{8}$$

where

c = the distance from the center of the Smith Chart to the center of the disk complement

r = the radius of the disk complement

Substituting the values for c and r from (7) into (8),

$$\left|\frac{S_{22}^* - S_{11}\Delta^*}{|S_{22}|^2 - |\Delta|^2}\right| - \frac{|S_{21}S_{12}|}{|S_{22}|^2 - |\Delta|^2} > 1$$

It is now important to note that the denominator of the expression for c must be positive since $|S_{22}|^2 - |\Delta|^2 > 0$. So one can simplify this expression as follows

$$\frac{|S_{22} - S_{11}^*\Delta| - |S_{21}S_{12}|}{|S_{22}|^2 - |\Delta|^2} > 1 \tag{9}$$

Case 2: $|S_{22}|^2 - |\Delta|^2 < 0$

In this case the range of the mapping "f^{-1}" is the region inside the disk defined by (6) and must be of the type illustrated in figure 1c. It is clear that USC is contained in this locus of points if and only if

$$r - c > 1 \tag{10}$$

Substituting the values for c and r from (6) into (10),

$$\frac{|S_{21}S_{12}|}{|\Delta|^2 - |S_{22}|^2} - \left|\frac{S_{22}^* - S_{11}\Delta^*}{|S_{22}|^2 - |\Delta|^2}\right| > 1$$

It is now important to note that the denominator of the expression for c must be negative since $|S_{22}|^2 - |\Delta|^2 < 0$. So one can simplify this expression as follows

$$\frac{|S_{21}S_{12}|}{|\Delta|^2 - |S_{22}|^2} - \frac{|S_{22}^* - S_{11}\Delta^*|}{|\Delta|^2 - |S_{22}|^2} > 1$$

By reordering the terms in the denominator, one arrives at the following result

$$\frac{|S_{22} - S_{11}^*\Delta| - |S_{21}S_{12}|}{|S_{22}|^2 - |\Delta|^2} > 1 \tag{11}$$

It is important to note that (11) is *identical* to (9), and thus a *single*, stability parameter emerges regardless of the value of $|S_{22}|^2 - |\Delta|^2$.

The apparent singularity presented by the denominator of (11) can be eliminated and the expression further simplified by noting that

$$\frac{|S_{22} - \Delta S_{11}^*|^2 - |S_{21}S_{12}|^2}{1 - |S_{11}|^2} = |S_{22}|^2 - |\Delta|^2 \tag{12}$$

Factoring the numerator of (12) which is the difference of two squares and substituting $|S_{22}|^2 - |\Delta|^2$ from (12) into (11) yields,

$$\mu \equiv \frac{1 - |S_{11}|^2}{|S_{22} - \Delta S_{11}^*| + |S_{21}S_{12}|} > 1. \tag{13}$$

It is interesting to note that the case where $|S_{22}|^2 - |\Delta|^2 = 0$ results in a stability circle which is a straight line but presents no difficulty with (13).

All steps taken above have been completely reversible, so it has been shown that the mapping "f^{-1}", illustrated in figures (1b) and (1c), will occur if and only if $\mu(S) > 1$.

3. PROOF THAT $\mu > 1 \Leftrightarrow$ UNCONDITIONAL STABILITY

In order to prove that $\mu > 1$ if and only if unconditional stability exists, it must first be shown that the mapping "f" illustrated in figure 1a implies that

$$K > 1 \tag{14a}$$

$$1 - |S_{22}|^2 > |S_{21}S_{22}| \tag{14b}$$

The range of the mapping of f ($|\Gamma_L| < 1$) is determined by $|f^{-1}(\Gamma_{in})| < 1$. Solving for Γ_{in}, one obtains the following,

$$|\Gamma_{in}|^2\left[|S_{22}|^2-1\right]+\Gamma_{in}\left[S_{11}^*-S_{22}\Delta^*\right]$$
$$+\Gamma_{in}^*\left[S_{11}-S_{22}^*\Delta\right]>|S_{11}|^2-|\Delta|^2 \quad (15)$$

It is now desirable to divide both sides of (15) by $|S_{22}|^2-1$. If $|S_{22}|^2-1>0$, the range of the mapping "f" would be a **disk complement** and contradictory to the assumption that the mapping is that illustrated in figure 1a. Therefore, $|S_{22}|^2-1<0$. Dividing (15) by $|S_{22}|^2-1$ and simplifying as illustrated in (4), one obtains

$$\left|\Gamma_{in}-\frac{S_{11}-S_{22}^*\Delta}{1-|S_{22}|^2}\right|<\frac{|S_{21}S_{12}|}{\left[1-|S_{22}|^2\right]} \quad (16)$$

By noting that this mapping results in a disc that lies inside the USC in the Γ_{in} plane (see figure 1a), we see that $c+r<1$ and moving r to the right hand side yields

$$\frac{|S_{11}-S_{22}^*\Delta|}{1-|S_{22}|^2}<1-\frac{|S_{21}S_{12}|}{1-|S_{22}|^2} \quad (17)$$

Since the left side of (17) is greater than or equal to zero, then

$$|S_{21}S_{12}|<1-|S_{22}|^2 \quad (18)$$

Furthermore, squaring both sides of (17) and substituting

$$|S_{11}-\Delta S_{22}^*|^2=|S_{21}S_{12}|^2+\left[1-|S_{22}|^2\right]\left[|S_{11}|^2-|\Delta|^2\right] \quad (19)$$

into the result yields

$$K\equiv\frac{1-|S_{22}|^2-|S_{11}|^2+|\Delta|^2}{2|S_{21}S_{12}|}>1 \quad (20)$$

All of the steps taken from (15) to (20) are completely reversible, so it has been shown that the mapping "f" illustrated in figure 1a implies that

$$K>1$$

$$1-|S_{22}|^2>|S_{21}S_{12}|$$

This is exactly the two conditions of (1) and (2e) which are known to be necessary and sufficient for unconditional stability of a linear 2-port. Thus it has been shown that $\mu>1$ if and only if a 2-port network is unconditionally stable.

We now look at the unilateral case of μ. It is clear by substitution that

$$\mu(unilateral)=\frac{1-|S_{11}|^2}{|S_{22}||1-|S_{11}|^2|} \quad (21)$$

so it is immediately obvious that $\mu>1$ if and only if $|S_{22}|<1$ and $|S_{11}|<1$, which are the necessary and sufficient conditions for unconditional stability of a unilateral circuit.

4. DEFINITION OF THE DUAL PARAMETER μ'

Another parameter, μ', can be defined based on the mapping function "g" in (3) and likewise $\mu'>1$ if and only if a 2-port network is unconditionally stable. The dual parameter is given by

$$\mu'\equiv\frac{1-|S_{22}|^2}{|S_{11}-\Delta S_{22}^*|+|S_{21}S_{12}|}>1 \quad (22)$$

This further implies that $\mu(S)>\Leftrightarrow\mu'(S)>1$.

5. CONCLUSIONS

It has been shown that a single parameter, μ, exists that is necessary and sufficient to show unconditional stability of any 2-port network. A companion parameter, μ', also exists and is necessary and sufficient to show unconditional stability of any 2-port as well. A comparison of the new stability parameter (μ or μ') for S-parameter values that satisfy or violate the traditional stability conditions (1) and (2) has been done and is left as an exercise for the reader.

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